One-ended spanning forests in hyperfinite graphs

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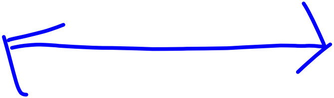
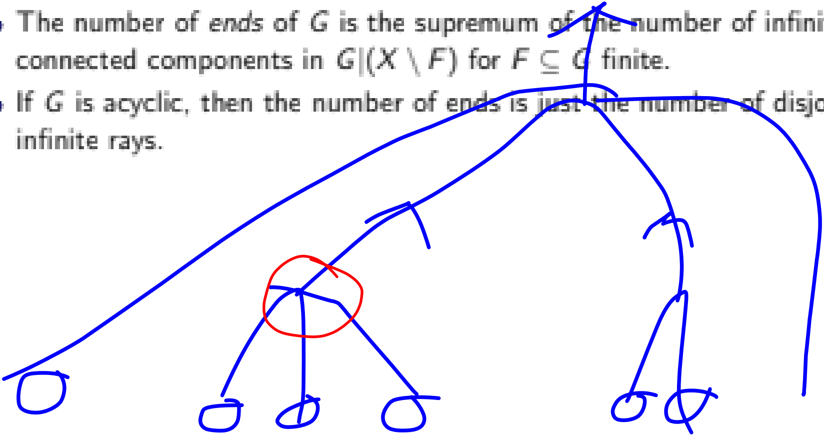
Let G be a (locally finite) connected graph on X

- The number of *ends* of G is the supremum of the number of infinite connected components in $G|(X \setminus F)$ for $F \subseteq G$ finite.



Let G be a (locally finite) connected graph on X

- The number of *ends* of G is the supremum of the number of infinite connected components in $G|(X \setminus F)$ for $F \subseteq G$ finite.
- If G is acyclic, then the number of ends is just the number of disjoint infinite rays.



Borel one-ended spanning subgraphs

Suppose G is a locally finite Borel graph on (X, μ).

- We say an acyclic borel $\mathcal{F} \subseteq G$ is an a.e. one-ended spanning forest if $V(\mathcal{F})$ is conull and each connected component of $\mathcal{F}$ is one-ended.

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Theorem (measurable Brooks)

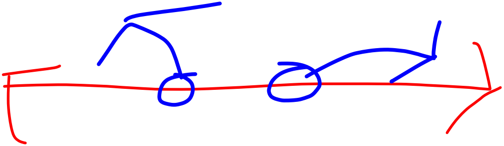
Suppose that G has max degree d and admits a Borel a.e. one-ended spanning forest. Then $\chi_\mu(G) \leq d$.

Characterizing graphs with Borel a.e. one-ended spanning forests

Conjecture

Suppose that E_G is a.e. non-smooth. TFAE:

- *G is a.e. not two-ended.*
- *G has a borel one-ended spanning forest a.e.*



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Theorem (CMT '16)

If G is acyclic and μ-nowhere 0 or 2 ended then it contains a borel a.e. spanning forest.

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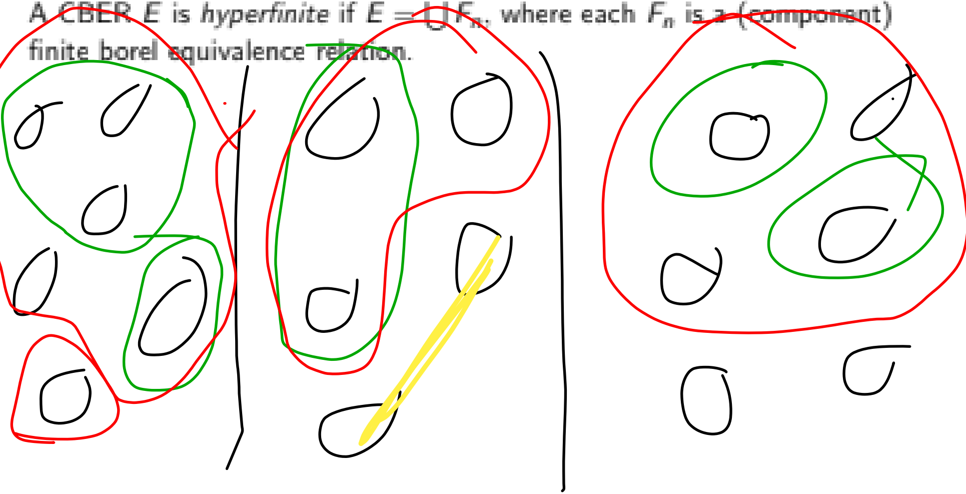
Theorem (CGMT '21)

Conjecture 1 is true for measure preserving graphs.

$$\mu(f(A)) = \mu(A)$$

A CBER E is *hyperfinite* if $E = \bigcup F_n$, where each F_n is a (component) finite borel equivalence relation.

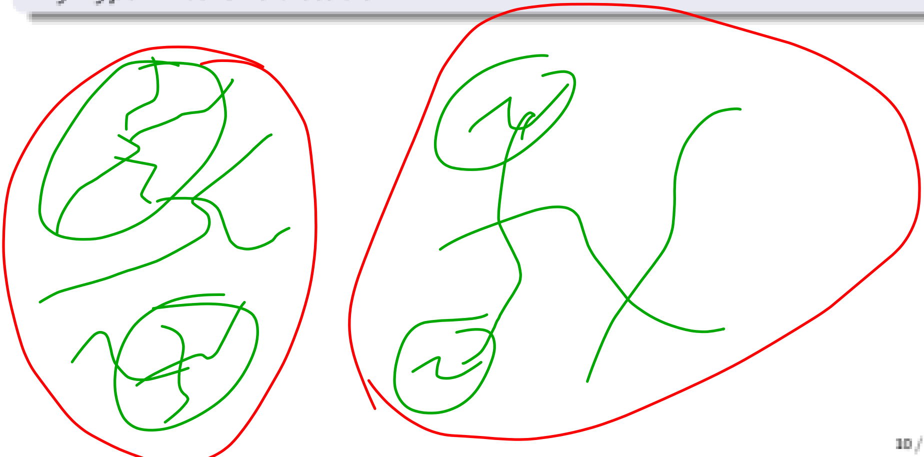


Spanning trees in hyperfinite pmp graphs

A CBER E is *hyperfinite* if $E = \bigcup F_n$, where each F_n is a (component) finite borel equivalence relation. $\nu(x) \le n$, 1-1 edges

Theorem

Any hyperfinite G is treeable.



Spanning trees in hyperfinite pmp graphs

A CBER E is *hyperfinite* if $E = \bigcup F_n$, where each F_n is a (component) finite borel equivalence relation.

Theorem

Any hyperfinite G is treeable.

Theorem

Any hyperfinite pmp G has a treeing with at most 2 ends a.e.

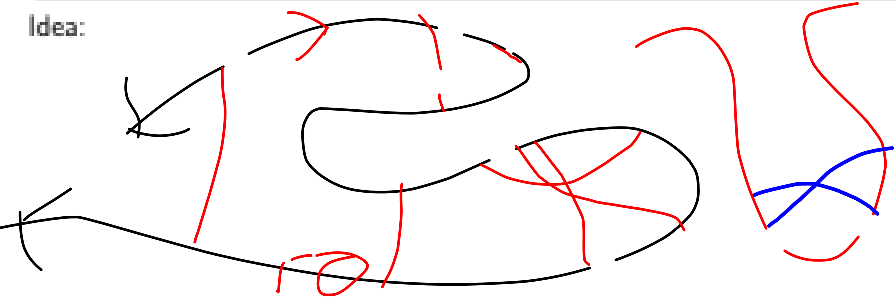
Q: $\int_A d(x) < \infty$

One-ended spanning trees

Theorem

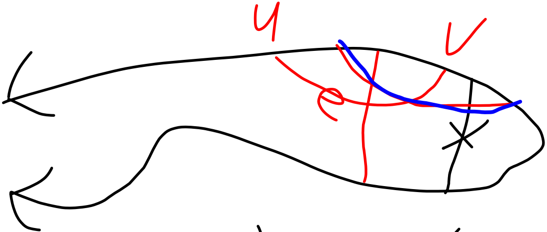
If G is hyperfinite, pmp, and almost nowhere 0 or 2 ended then G has an a.e. one-ended treeing.

Idea:



The details

- For $e \in G$ let $[e]$ be the vertices (strictly) on the T path between the endpoints of e.
- Let $n(e) = max\{|[e']| : e' = (u, v) \text{ with } u \in [e] \text{ or } v \in [e]\}$.
- For $G' \subseteq G$, let $[G'] = \bigcup_{e \in G'} [e]$. Let $G_N = \{e : |[e]|, n(e) \leq N\}$.



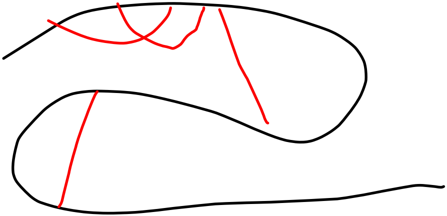
H_n: $V(H_n) = E(G_N) = G_n$

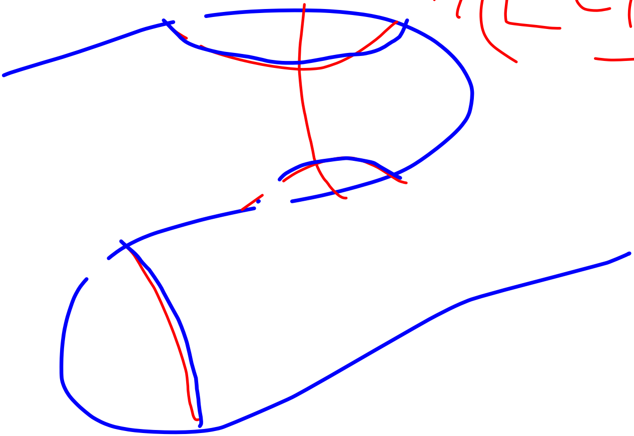
$E(H_n) = \{(e, f) : [e] \cap [f]$

$$G'_0 = G_1$$

$$G'_{i+1} = \{ G'_i - e \mid e \in C_i : [e] \subseteq [G'_i - e] \}$$

$$[G'_k] = [G_1]$$





$$\mu\left(\bigcup C_i\right) \geq \frac{\mu(H_1)}{3}$$

$$\geq \frac{\mu\left(\bigcup H_1\right)}{3}$$

dropping the measure preserving

Theorem

Any hyperfinite almost nowhere 0 or 2-ended G contains a Borel a.e. one-ended spanning forest

dropping the measure preserving

Theorem

Any hyperfinite almost nowhere 0 or 2-ended G contains a Borel a.e. one-ended spanning forest

Question

Does every one-ended hyperfinite G contain an a.e./generic one-ended treeing?

dropping the measure preserving

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Question

Does every one-ended hyperfinite G contain an a.e./generic one-ended treeing?

References

- Anton Bernshteyn, A proof of the Kechris-Solecki-Todorcevic dichotomy, preprint.
- A. S. Kechris, S. Solecki, and S. Todorcevic, Borel chromatic numbers, Advances in Mathematics141(1999), no. 1, 1-44.
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